GEOMETRY OF THE INTERSECTION RING AND VANISHING RELATIONS IN THE COHOMOLOGY OF THE MODULI SPACE OF PARABOLIC BUNDLES ON A CURVE

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ABSTRACT. We study the ring generated by the Chern classes of tautological line bundles on the moduli space of parabolic bundles of arbitrary rank on a Riemann surface. We show the Poincaré duals to these Chern classes have simple geometric representatives. We use this construction to show that the ring generated by these Chern classes vanishes below the dimension of the moduli space, in analogy with the Newstead-Ramanan conjecture for stable bundles.

1. INTRODUCTION

Let G be a compact Lie group with maximal torus T, let Σ^g be a compact, connected, oriented 2-manifold of genus g, and let $p \in \Sigma^g$. For $t \in T$, let

$$R_g(t) = \{(A_1, \ldots, A_g, B_1, \ldots, B_g) \in G^{2g} | \prod_{i=1}^{g}[A_i, B_i] \sim t\}$$

(where $\sim$ denotes conjugacy in G). The fundamental group $\pi_1(\Sigma^g \setminus \{p\})$ can be presented by generators $a_1, \ldots, a_g, b_1, \ldots, b_g, c$ with the relation $\prod_{i=1}^{g}[a_i, b_i] = c$, where c can be thought of as representing the boundary curve of a small disc containing p; we choose such a set of generators. Then

$$R_g(t) = \{\rho \in Hom(\pi_1(\Sigma^g \setminus \{p\}), SU(N)) | \rho(c) \sim t\},$$

G acts on $R_g(t)$ by conjugation, and $S_g(t) = R_g(t)/G$ is the space of characters of the fundamental group of $\Sigma^g \setminus \{p\}$ in G where the conjugacy class of the image of c is fixed.

When $t = e$, we get the moduli space $\bar{S}_g = Hom(\pi_1(\Sigma^g), G)/G$ of flat connections on Σ^g.

Take $t = \xi \in Z(G)$ to be in the centre of G. In particular, take $G = SU(N)$ and $\xi = e^{2\pi i k/N} I \in Z(G)$, where $(k, N) = 1$. Then the space

$$R_g(\xi) = \{(A_1, \ldots, A_g, B_1, \ldots, B_g) \in G^{2g} | \prod_{i=1}^{g}[A_i, B_i] = \xi\}$$

is the moduli space of flat connections on a principal G-bundle over Σ^g. If we equip Σ^g with a conformal structure, then $S_g(\xi)$ acquires a Kähler structure as a moduli space of stable rank N vector bundles, of degree k and fixed determinant, over the corresponding Riemann surface. In this case, the following generalisations of the Newstead-Ramanan conjecture have been established:

Theorem 1.1 ([32]). *The Chern classes of $S_g(\xi)$ vanish above degree $N(N-1)(g-1)$.*

Theorem 1.2 ([7]). *The ring generated by the Chern classes of the vector bundle associated to $R_g(\xi) \to S_g(\xi)$ via the standard representation of $SU(N)$ on $\mathbb{C}^N$ vanishes in dimension strictly greater than $2N(N-1)(g-1)$.*

Let $a_2, \ldots, a_r$ be the Chern classes of the vector bundle associated to $R_g(\xi)$ via the standard representation of $SU(N)$ on $\mathbb{C}^N$. Earl shows in [6] that the Pontryagin ring of $S_g(\xi)$ is contained in the subring of $H^*(S_g(\xi))$ generated by the a_i, and so Theorem 1.2 implies

Theorem 1.3 ([7]). *The Pontryagin ring of $S_g(\xi)$ vanishes in dimension strictly greater than $2N(N-1)(g-1)$.*

These results are generalisations of a conjecture of Newstead published in [27]. For some other references on this subject, see [1, 2, 3, 4, 5, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 28, 29, 30, 31, 33, 35, 36, 37].

We now take t to be a generic element of T:

Definition 1.4. Let $t_1, \ldots, t_N \in (0,1)$ be such that $t_1 + \cdots + t_N \in \mathbb{Z}$, but the sum of any proper nonempty subset of the t_j is not an integer, and let $t = \mathrm{Diag}(e^{2\pi i t_1}, \ldots, e^{2\pi i t_N}) \in T$.

Consider

$$R_g(t) = \{(A_1, \ldots, A_g, B_1, \ldots, B_g) \in G^{2g} | \prod_{i=1}^{g} [A_i, B_i] \sim t\}.$$

Again, G acts on $R_g(t)$ by conjugation; in this case, $S_g(t) = R_g(t)/G$ is a moduli space of rank N vector bundles over Σ^g with parabolic structure at the marked point p.

Consider the torus bundle $V_g(t) \to S_g(t)$ given by

$$V_g(t) = \{(A_1, \ldots, A_g, B_1, \ldots, B_g) \in G^{2g} | \prod_{i=1}^{g} [A_i, B_i] = t\}$$

$$= \{\rho \in Hom(\pi_1(\Sigma^g \setminus \{p\}), SU(N)) | \rho(c) = t\};$$

then $S_g(t) = R_g(t)/G = V_g(t)/T$. For $1 \le i, j \le N$ with $i \ne j$, let $L_{ij} \to S_g(t)$ be the line bundle associated to $V_g(t)$ by the representation

(1)
$$T \times \mathbb{C} \to \mathbb{C}$$
$$(\mathrm{Diag}(e^{\sqrt{-1}\theta_1}, \ldots, e^{\sqrt{-1}\theta_N}), z) \mapsto e^{\sqrt{-1}(\theta_i - \theta_j)} z.$$

We will denote this representation of T by $\mathbb{C}_{(ij)}$, and its weight by χ_{ij}. Note that since $L_{ij} \otimes L_{jk} = L_{ik}$ and $L_{ij} = L_{ji}^*$, for i, j, k all distinct, we have $c_1(L_{ij}) + c_1(L_{jk}) = c_1(L_{ik})$ and $c_1(L_{ij}) = -c_1(L_{ji})$. We are interested in the subring of $H^*(S_g(t); \mathbb{Q})$ generated by the $c_1(L_{ij})$. When $G = SU(2)$, $V_g(t)$ is a circle bundle, and

Theorem 1.5 ([34]). $c_1(L_{12})^{2g} = 0.$

We will sketch here the proof of Theorem 1.5 found in [34], as the purpose of this paper is to extend this technique to arbitrary rank. The idea is to find explicit geometric cycles Poincaré dual to the Chern class $c_1(V_g(t))$. For $1 \le i \le g$, consider the sections s_{A_i} of $L_{12} \to S_g(t)$ induced by the equivariant maps

$$V_g(t) \to \mathbb{C}_{(12)}$$
$$(A_1, \ldots, A_g, B_1, \ldots, B_g) \mapsto (A_i)_{12},$$

where the subscript 12 denotes the $(1,2)$ matrix entry. These sections s_{A_i} vanish on the cycles

$$D(A_i) = \{(A_1, \ldots, A_g, B_1, \ldots, B_g) \in V_g(t) | [A_i, t] = 1\}/T.$$

Define sections s_{B_i} and cycles $D(B_i)$ similarly, and consider the intersection
$$D := D(A_1) \cap \cdots \cap D(A_g) \cap D(B_1) \cap \cdots \cap D(B_g).$$
This is the image in $S_g(t)$ of the set of elements $(A_1, \ldots, A_g, B_1, \ldots, B_g) \in V_g(t)$ with the matrices $A_1, \ldots, A_g, B_1, \ldots, B_g$ all diagonal, so $\prod_{i=1}^{g}[A_i . B_i] = 1$. Since $t \neq 1$ by Definition 1.4, $D = \emptyset$, and so $c_1(L_{12})^{2g} = 0$.

The purpose of this paper is to generalise this result to $G = SU(N)$. Our theorem is

Theorem 1.6. *For $1 \leq i, j \leq N$ and $i \neq j$, let k_{ij} be nonnegative integers. Then the cohomology class*
$$\prod_{\substack{1 \leq i,j \leq N \\ i \neq j}} c_1(L_{ij})^{k_{ij}} \in H^*(S_g(t); \mathbb{Q})$$
vanishes whenever
$$\sum_{\substack{1 \leq i,j \leq N \\ i \neq j}} k_{ij} \geq N(N-1)g - N + 2.$$

Remark 1.7. The dimension of $S_g(t)$ is given by
$$\begin{aligned}
\dim(S_g(t)) &= 2g \dim SU(N) - \dim T - \dim SU(N) \\
&= 2g(N^2 - 1) - (N - 1) - (N^2 - 1) \\
&= 2g(N^2 - 1) - N^2 - N + 2.
\end{aligned}$$
Theorem 1.6 above says that monomials in the $c_1(L_{ij})$ vanish in degree r for $r \geq 2gN(N-1) - 2N + 4$, which is well below the dimension of $S_g(t)$.

The proof uses the same technique as [34] but the combinatorics of the intersections is much more complicated; we illustrate it here for the case $G = SU(3)$.

For $1 \leq i, j \leq 3$ with $i \neq j$, consider the line bundles $L_{ij} \to S_g(t)$ associated to $V_g(t)$ as above. Their Chern classes $c_{ij} = c_1(L_{ij})$ satisfy
$$\begin{aligned}
c_{ij} + c_{jk} + c_{ki} &= 0 \\
c_{ij} + c_{ji} &= 0.
\end{aligned}$$
(2)

For $1 \leq m \leq g$, we can as before find sections $s_{A_m}^{ij}$ of $L_{ij} \to S_g(t)$ that are zero on
$$D_{ij}(A_m) = \{(A_1, \ldots, A_g, B_1, \ldots, B_g) \in V_g(t) | (A_m)_{ij} = 0\}/T,$$
and similarly sections $s_{B_m}^{ij}$ with zero locus $D_{ij}(B_m)$. The intersection
$$D := \bigcap_{m=1}^{g} D_{12}(A_m) \cap D_{12}(B_m) \cap D_{13}(A_m) \cap D_{13}(B_m)$$
is the image in $S_g(t)$ of the set of elements of $V_g(t)$ where $(A_m)_{12} = (A_m)_{13} = 0$ and $(B_m)_{12} = (B_m)_{13} = 0$ for all m. Hence $(\prod_{m=1}^{g}[A_m, B_m])_{11} = 1$, so again if t is generic as in Definition 1.4, $D = \emptyset$. Hence the monomial $c_{12}^{2g} c_{13}^{2g}$ vanishes, and similarly, so do $c_{23}^{2g} c_{21}^{2g}$ and $c_{31}^{2g} c_{32}^{2g}$. But any monomial in the c_{ij} of degree at least $6g - 1$ may be written using the relations (2) as a sum of monomials containing at least one such factor, and thus

Proposition 1.8. *Let $G = SU(3)$. Then any monomial in the Chern classes of the L_{ij} of degree at least $6g - 1$ vanishes.*

The generalisation of this argument to higher rank requires more careful attention to the combinatorics and algebra of possible monomials.

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2. COMBINATORIAL PRELIMINARIES

We fix an integer $g \geq 2$; in the geometric application this is the genus of the 2-manifold Σ^g.

Notation. For any set X, we will denote by $X^{(2)}$ the set of 2-element subsets of X.

Lemma 2.1. *Let X and Y be finite sets with $|X| = n \geq 3$ and $|Y| \geq n(n-1)g - n + 2$, and let $f : Y \to X^{(2)}$ be a function. Then there exists $z \in X$ such that $|f^{-1}((X \setminus \{z\})^{(2)})| \geq (n-1)(n-2)g - n + 3$.*

Proof. For $x \in X$, let Q_x denote the set $(X \setminus \{x\})^{(2)}$. Suppose there is no z for which $|f^{-1}(Q_z)| \geq (n-1)(n-2)g - n + 3$. So for each $x \in X$, $|f^{-1}(Q_x)| \leq (n-1)(n-2)g - n + 2$, and thus

$$\sum_{x \in X} |f^{-1}(Q_x)| \leq n(n-1)(n-2)g - n(n-2).$$

But each element $\{u, v\} \in X^{(2)}$ is contained in exactly $n - 2$ of the sets Q_x. So $\sum_{x \in X} |f^{-1}(Q_x)| = (n-2)|Y|$. Hence

$$|Y| \leq \frac{n(n-1)(n-2)g - n(n-2)}{n-2}$$
$$= n(n-1)g - n$$
$$< |Y|.$$

This is a contradiction, so such a z must exist. $\qquad\qquad\square$

Definition 2.2. Let X be a finite set. A *block* in $X \times X$ is a subset $B \subset X \times X$ of the form $V \times (X \setminus V)$, where $V \subset X$ is a proper nonempty subset. If $|V| = h$ we call B an *h-by-$(|X| - h)$ block*.

Definition 2.3. For a finite set X, let $\mathcal{B}[X]$ be the set of blocks in $X \times X$.

Notation. If $n \in \mathbb{N}$, we will write $[n]$ for the set $\{1, \ldots, n\}$.

Remark 2.4. Suppose $A \in SU(N)$ and $1 \leq h \leq N$, let $V \subset [N]$ be a subset with $|V| = h$, and consider the block $B = V \times ([N] \setminus V) \subset [N] \times [N]$. If $A_{ij} = 0$ for all $(i, j) \in B$, then there is some ordering of basis elements for which A is upper block diagonal. More precisely, if σ is a permutation of $[N]$ such that $\sigma(V) = [h]$, then the matrix $(C_{ij}) = (A_{\sigma^{-1}(i)\sigma^{-1}(j)})$ has the form

$$\left(\begin{array}{ccc|ccc}
A_{\sigma^{-1}(1)\sigma^{-1}(1)} & \cdots & A_{\sigma^{-1}(1)\sigma^{-1}(h)} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
A_{\sigma^{-1}(N-h)\sigma^{-1}(1)} & \cdots & A_{\sigma^{-1}(N-h)\sigma^{-1}(h)} & 0 & \cdots & 0 \\
\hline
| & & | & | & & | \\
u_1 & \cdots & u_h & v_1 & \cdots & v_{N-h} \\
| & & | & | & & |
\end{array} \right).$$

Since A, and hence C, are unitary, the vectors v_i form a basis for $\mathbb{C}^{N-h}$, and $u_i \cdot v_j = 0$ for all i, j. Hence each of the u_i must be zero, so the matrix C is block diagonal. Hence $A_{ji} = 0$ for all $(i, j) \in B$. Thus the condition $A_{ij} = 0$ for all $(i, j) \in B$ implies that A is block diagonal, with blocks of size h and $N - h$, up to reordering of basis elements.

3. ALGEBRAIC PRELIMINARIES

Let X be a finite set.

Definition 3.1. Let $\mathbb{Q}[x_{ij}]$ be the ring $\mathbb{Q}[\{x_{ij} | 1 \leq i, j \leq |X|, i \neq j\}]$, where we adjoin variables x_{ij} for all ordered pairs (i, j) with $i, j \in X$ and $i \neq j$.

We will need the following lemmas.

Lemma 3.2. *Suppose $|X| = n \geq 3$, and let $p \in \mathbb{Q}[x_{ij}]$ be a monomial of degree at least $n(n-1)g - n + 2$. Then there exists some $z \in X$ such that if we factor p as $p = qr$, where $q \in \mathbb{Q}[\{x_{iz}, x_{zi} | i \in X \setminus \{z\}\}]$ and $r \in \mathbb{Q}[\{x_{ij} | i, j \in X \setminus \{z\}, i \neq j\}]$ are monomials, then r has degree at least $(n-1)(n-2)g - n + 3$.*

Proof. Write $p = \lambda \prod_{i,j} x_{ij}^{d_{ij}}$. Let Y_{ij} be disjoint sets with $|Y_{ij}| = d_{ij}$, for each pair (i,j) with $i, j \in X$ and $i \neq j$. Let $Y = \cup Y_{ij}$. We have

$$|Y| = \sum_{i,j} d_{ij} \geq n(n-1)g - n + 2.$$

Consider the function

$$f : Y \to X^{(2)}$$

given by

$$f|_{Y_{ij}} = \{i, j\}.$$

Then

$$|f^{-1}(\{i, j\})| = d_{ij} + d_{ji}$$

for all $\{i, j\} \in X^{(2)}$. By Lemma 2.1, there exists $z \in X$ with

$$|f^{-1}((X \setminus \{z\})^{(2)})| \geq (n-1)(n-2)g - n + 3.$$

But

$$|f^{-1}((X \setminus \{z\})^{(2)})| = \sum_{\{i,j\} \in (X \setminus \{z\})^{(2)}} d_{ij} + d_{ji}$$

$$= \deg(r),$$

and so $\deg(r) \geq (n-1)(n-2)g - n + 3$. $\qquad\square$

Lemma 3.3. *Let $z \in X$ and let $w, h \in \mathbb{N}$ with $w + h = |X| - 1$. Let $\eta \in \mathbb{Q}[\{x_{zi} | i \in X \setminus \{z\}\}]$ be a monomial of degree at least $|X|(|X|-1)g - |X| + 2 - 2gwh$. Given a partition $X \setminus \{z\} = \{e_1, \ldots, e_h\} \sqcup \{f_1, \ldots, f_w\}$, factor η as $\eta = \eta_h \eta_w$, with $\eta_h \in \mathbb{Q}[\{x_{ze_i} | 1 \leq i \leq h\}]$ and $\eta_w \in \mathbb{Q}[\{x_{zf_i} | 1 \leq i \leq w\}]$. Then either the degree of η_h is at least $gh(h+1) - h + 1$, or the degree of η_w is at least $gw(w+1) - w + 1$.*

Proof. Suppose $\deg(\eta_h) < gh(h+1) - h + 1$. Then

$$\begin{aligned}
\deg(\eta_w) &= \deg(\eta) - \deg(\eta_h) \\
&\geq |X|(|X|-1)g - |X| + 2 - 2gwh - gh(h+1) + h \\
&= (w+h+1)(w+h)g - (w+h+1) + 2 - 2gwh - gh(h+1) + h \\
&= g(w^2 + 2hw + h^2 + w + h - 2wh - h^2 - h) - w - h + 1 + h \\
&= gw(w+1) - w + 1. \qquad\square
\end{aligned}$$

Definition 3.4. Let $I \subset \mathbb{Q}[x_{ij}]$ be the ideal generated by $x_{ij} + x_{ji}$ and $x_{ij} + x_{jk} + x_{ki}$, for all triples of distinct elements $i, j, k \in X$. Let $R = \mathbb{Q}[x_{ij}]/I$ be the quotient of $\mathbb{Q}[x_{ij}]$ by this ideal.

Note that the quotient preserves the grading by degree. If $\zeta \in \mathbb{Q}[x_{ij}]$ we will write $[\zeta]$ for its image in R.

Lemma 3.5. *Let $\xi \in \mathbb{Q}[x_{ij}]$ be a monomial, and let $z \in X$. Then there exists a homogeneous polynomial $\eta \in \mathbb{Q}[\{x_{zj} | j \in X \setminus \{z\}\}]$ of the same degree as ξ such that $[\xi] = [\eta] \in R$.*

Proof. If $\xi = \lambda \prod_{i,j} x_{ij}^{d_{ij}}$ where $\lambda \in \mathbb{Q}$, let

$$\eta = \lambda \prod_{i,j \neq z} (-x_{zi} + x_{zj})^{d_{ij}} \prod_{i \neq z} (-1)^{d_{iz}} x_{zi}^{d_{iz}} \prod_{j \neq z} x_{zj}^{d_{zj}}.$$

Then $[\xi] = [\eta]$, and the degree of each term of η is equal to the degree of ξ. $\qquad\square$

Proposition 3.6. *Let X be a finite set with $|X| \geq 2$. Let $R = \mathbb{Q}[x_{ij}]/I$ as before. Let $\zeta \in \mathbb{Q}[x_{ij}]$ be a monomial of degree at least $|X|(|X| - 1)g - |X| + 2$. Then for each block $B \subset X \times X$ we can find a monomial ψ_B in $\mathbb{Q}[x_{ij}]$ such that*

$$[\zeta] = \left[\sum_{B \in \mathcal{B}[X]} \psi_B \prod_{(i,j) \in B} x_{ij}^{2g} \right].$$

Proof. By induction on $|X|$.

If $|X| = 2$, take $X = \{1, 2\}$. The monomials in $\mathbb{Q}[x_{ij}]$ of degree at least $2g$ are of the form $\lambda x_{12}^a x_{21}^b$ where $a + b \geq 2g$, $\lambda \in \mathbb{Q}$. The set $\{(1,2)\}$ is a 1-by-1 block in $X \times X$. Since $[x_{12}] = [-x_{21}]$ in R, the class $[\lambda x_{12}^a x_{21}^b] = [\lambda(-1)^b x_{12}^{a+b-2g} x_{12}^{2g}] \in R$ is of the desired form.

Now suppose $|X| = n \geq 3$. Let $\zeta \in \mathbb{Q}[x_{ij}]$ be a monomial of degree $d \geq n(n-1)g - n + 2$. By Lemma 3.2, there exists $z \in X$ such that if we factor ζ as $\zeta = qr$, where $q \in \mathbb{Q}[\{x_{iz}, x_{zi} | i \in X \setminus \{z\}\}]$ and $r \in \mathbb{Q}[\{x_{ij} | i, j \in X \setminus \{z\}, i \neq j\}]$ are monomials, then the degree of r is at least $(n-1)(n-2)g - n + 3$.

By the inductive hypothesis, for each block $C \subset (X \setminus \{z\}) \times (X \setminus \{z\})$ we can find a monomial $\theta_C \in \mathbb{Q}[\{x_{ij} | i, j \in X \setminus \{z\}, i \neq j\}]$ such that

$$[r] = \left[\sum_{C \in \mathcal{B}[X \setminus \{z\}]} \theta_C \prod_{(i,j) \in C} x_{ij}^{2g} \right],$$

and so

$$[\zeta] = [qr] = \left[q \sum_{C \in \mathcal{B}[X \setminus \{z\}]} \theta_C \prod_{(i,j) \in C} x_{ij}^{2g} \right].$$

It suffices to show that each nonzero monomial in the sum can be written as a sum of terms having the desired form. For each C with $\theta_C \neq 0$, consider

$$\left[q \theta_C \prod_{(i,j) \in C} x_{ij}^{2g} \right].$$

This is a monomial of degree $d \geq n(n-1)g - n + 2$. Suppose the block C is given by

$$C = \{e_1, \ldots, e_h\} \times \{f_1, \ldots, f_w\},$$

where $h, w \geq 1$ and $X \setminus \{z\}$ is the disjoint union $X \setminus \{z\} = \{e_1, \ldots, e_h\} \sqcup \{f_1, \ldots, f_w\}$ (so $w + h = n - 1$). By Lemma 3.5, we can find a homogeneous polynomial $p_1 + \cdots + p_m$, where $p_1, \ldots, p_m$ are monomials in $\mathbb{Q}[\{x_{zj} | j \in X \setminus \{z\}\}]$, such that

$$\left[q \theta_C \prod_{(i,j) \in C} x_{ij}^{2g} \right] = \left[(p_1 + \cdots + p_m) \prod_{(i,j) \in C} x_{ij}^{2g} \right].$$

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Again, it suffices to show that each monomial in the sum can be written as a sum of terms having the desired form, so consider

$$\left[p \prod_{(i,j)\in C} x_{ij}^{2g} \right],$$

where $p \in \{p_1, \ldots, p_m\}$. Note that

$$\deg(p) = d - 2gwh \geq n(n-1)g - n + 2 - 2gwh.$$

Factor p as $p = p_h p_w$ where $p_h \in \mathbb{Q}[\{x_{ze_i}|1 \leq i \leq h\}]$ and $p_w \in \mathbb{Q}[\{x_{zf_j}|1 \leq j \leq w\}]$ are monomials. By Lemma 3.3, either $\deg(p_h) \geq gh(h+1) - h + 1$ or $\deg(p_w) \geq gw(w+1) - w + 1$; without loss of generality we assume the former.

By the inductive hypothesis, for each block $D \subset \{e_1, \ldots, e_h, z\} \times \{e_1, \ldots, e_h, z\}$ we can find a monomial $\phi_D \in \mathbb{Q}[\{x_{ij}|i,j \in \{e_1, \ldots, e_h, z\}, i \neq j\}]$ such that

$$[p_h] = \left[\sum_{D \in \mathcal{B}[\{e_1, \ldots, e_h, z\}]} \phi_D \prod_{(i,j)\in D} x_{ij}^{2g} \right],$$

and so

$$\left[p \prod_{(i,j)\in C} x_{ij}^{2g} \right] = \left[p_w \sum_{D \in \mathcal{B}[\{e_1, \ldots, e_h, z\}]} \phi_D \prod_{(i,j)\in D} x_{ij}^{2g} \prod_{(i,j)\in C} x_{ij}^{2g} \right].$$

For each D, consider the monomial

$$\left[p_w \phi_D \prod_{(i,j)\in D} x_{ij}^{2g} \prod_{(i,j)\in C} x_{ij}^{2g} \right].$$

Observe that $C \cap D = \emptyset$, and so

$$\left[p_w \phi_D \prod_{(i,j)\in D} x_{ij}^{2g} \prod_{(i,j)\in C} x_{ij}^{2g} \right] = \left[p_w \phi_D \prod_{(i,j)\in C \cup D} x_{ij}^{2g} \right].$$

We may assume

$$D = \{e_{\sigma(1)}, \ldots, e_{\sigma(d)}\} \times \{e_{\sigma(d+1)}, \ldots, e_{\sigma(h)}, z\},$$

for some permutation σ of $[h]$ and $1 \leq d \leq h$, since $x_{ij} - x_{ji} \in I$. But

$$C \cup D = \{e_1, \ldots, e_h\} \times \{f_1, \ldots, f_w\} \cup \{e_{\sigma(1)}, \ldots, e_{\sigma(d)}\} \times \{e_{\sigma(d+1)}, \ldots, e_{\sigma(h)}, z\}$$

contains

$$E := \{e_{\sigma(1)}, \ldots, e_{\sigma(d)}\} \times \{e_{\sigma(d+1)}, \ldots, e_{\sigma(h)}, f_1, \ldots, f_w, z\} \in \mathcal{B}[X].$$

So

$$\left[p_w \phi_D \prod_{(i,j)\in D} x_{ij}^{2g} \prod_{(i,j)\in C} x_{ij}^{2g} \right] = \left[\psi_E \prod_{(i,j)\in E} x_{ij}^{2g} \right]$$

for some $\psi_E \in \mathbb{Q}[x_{ij}]$. We have shown that $[\zeta]$ has a representative in $\mathbb{Q}[x_{ij}]$ that is a sum of monomials of this form, i.e.

$$[\zeta] = \left[\sum_{B \in \mathcal{B}[X]} \psi_B \prod_{(i,j)\in B} x_{ij}^{2g} \right],$$

for some monomials ψ_B in $\mathbb{Q}[x_{ij}]$. $\qquad \square$

4. Proof of the main theorem

Definition 4.1. Suppose $u \in \{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ is one of the chosen generators of $\pi_1(\Sigma^g \setminus \{p\})$, and define maps

$$f_u^{ij} : V_g(t) \to \mathbb{C}_{(ij)}$$
$$\rho \mapsto (\rho(u))_{ij}$$

for each pair $(i.j)$ with $1 \leq i, j \leq N$ and $i \neq j$. These maps f_u^{ij} are T-equivariant since T acts on the matrix entry $(\rho(u))_{ij}$ with weight χ_{ij}. These maps then induce sections

$$s_u^{ij} : S_g(t) \to V_g(t) \times_T \mathbb{C}_{(ij)}$$
$$[\rho] \mapsto (\rho, f_u^{ij}(\rho))$$

of the line bundles L_{ij}.

Let D_u^{ij} be the image in $S_g(t)$ of the subspace $\{\rho \in V_g(t) | (\rho(u))_{ij} = 0\}$. Then the section s_u^{ij} is nonzero on the complement of D_u^{ij}.

To prove Theorem 1.6, we will show that intersections of certain sets of these subspaces D_u^{ij} are empty, and conclude that the corresponding polynomials in the Chern classes $c_1(L_{ij})$ are zero. This is the same technique that was used in [34].

Lemma 4.2. *Let M be a manifold. Let $\mathcal{L}_i \to M$, $i = 1, \ldots, m$, be complex line bundles with sections $s_i : M \to \mathcal{L}_i$. If these sections have no common zeros, i.e. $s_1^{-1}(0) \cap \cdots \cap s_m^{-1}(0) = \emptyset$, then $c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_m) = 0$.*

Proof. Consider the vector bundle $E := \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_m \to M$ with the section

$$\sigma := (s_1, \cdots, s_m) : M \to E.$$

The section σ is nowhere zero, so the Euler class $e(E) = 0$. Since $c_m(E) = e(E)$, the top Chern class $c_m(E) = c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_m)$ is equal to zero. $\square$

Lemma 4.3. *Let $0 < h < N$, let $B \subset [N] \times [N]$ be an h-by-$N - h$ block, and let*

$$\zeta = \prod_{(i,j) \in B} c_1(L_{ij})^{2g} \in H^{4gh(N-h)}(S_g(t)).$$

Then $\zeta = 0$.

Proof. Consider the sections $s_u^{ij} : S_g(t) \to L_{ij}$ (as defined in 4.1), for $(i,j) \in B$ and generators $u \in \{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ of $\pi_1(\Sigma^g \setminus \{p\})$. We have $(s_u^{ij})^{-1}(0) = D_u^{ij}$. By Lemma 4.2, it suffices to show that

$$D := \bigcap_{(i,j) \in B} (D_{a_1}^{ij} \cap \cdots \cap D_{a_g}^{ij} \cap D_{b_1}^{ij} \cap \cdots \cap D_{b_g}^{ij}) = \emptyset.$$

By definition, D is the image in $S_g(t)$ of the set of homomorphisms $\rho \in V_g(t)$ such that the $(i,j)^{th}$ entry of $\rho(u)$ is zero for all $(i,j) \in B$ and all $u \in \{a_1, \ldots, a_g, b_1, \ldots, b_g\}$. Suppose $\rho \in D$. By Remark 2.4, we can find a permutation σ of $[N]$ such that the matrices

$$(\Phi(u))_{ij}) := (\rho(u))_{\sigma^{-1}(i)\sigma^{-1}(j)},$$

for $u \in \{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ are all block diagonal with blocks of size h and $N-h$. So $\prod_{i=1}^{g} [\Phi(a_i), \Phi(b_i)]$ is also block diagonal with blocks of size h and $N - h$, and each block has determinant equal to 1. Let $E \in SU(N)$ be a product of elementary matrices representing this permutation σ of basis elements, so that

$$\rho(u) = E^\dagger \Phi(u) E \text{ for all } u \in \{a_1, \ldots, a_g, b_1, \ldots, b_g\}.$$

In particular, the matrix $EtE^\dagger$ is obtained from $t \in T$ by permuting the diagonal entries. Then

$$t = \rho(c) = \prod_{i=1}^{g}[\rho(a_i), \rho(b_i)]$$

$$= \prod_{i=1}^{g}[E^\dagger \Phi(a_i)E, E^\dagger \Phi(b_i)E]$$

$$= \prod_{i=1}^{g} E^\dagger[\Phi(a_i), \Phi(b_i)]E$$

$$= E^\dagger \left(\prod_{i=1}^{g}[\Phi(a_i), \Phi(b_i)] \right) E.$$

Thus $E\rho(c)E^\dagger = EtE^\dagger = \prod_{i=1}^{g}[\Phi(a_i), \Phi(b_i)]$ is a diagonal matrix where the first h diagonal entries have product equal to 1. But this is impossible because for $\rho \in V_g(t)$ we chose $t = \rho(c)$ such that no h of its diagonal entries could have product equal to 1 (see Definition 1.4). Hence the set D of such ρ is empty. $\qquad\square$

Proof of Theorem 1.6. Let $X = [N]$ and consider the rings $\mathbb{Q}[x_{ij}]$ and R as in Definitions 3.1 and 3.4. Let $J \subset H^*(S_g(t))$ be the subring generated by the $c_1(L_{ij})$ for $1 \leq i, j \leq N$ and $i \neq j$. Since $c_1(L_{ij}) = -c_1(L_{ji})$ and $c_1(L_{ij}) + c_1(L_{jk}) = c_1(L_{ik})$, the map

$$\pi : R \twoheadrightarrow J$$

$$[x_{ij}] \mapsto c_1(L_{ij})$$

defines a ring homomorphism. Consider the element

$$\prod_{\substack{1 \leq i,j \leq N \\ i \neq j}} c_1(L_{ij})^{k_{ij}} \in J.$$

It has a representative

$$\left[\prod_{\substack{1 \leq i,j \leq N \\ i \neq j}} x_{ij}^{k_{ij}} \right]$$

in R. Suppose

$$\sum_{\substack{1 \leq i,j \leq N \\ i \neq j}} k_{ij} \geq N(N-1)g - N + 2.$$

Then by Proposition 3.6, for each block $B \subset [N] \times [N]$ we can find a monomial θ_B in $\mathbb{Q}[x_{ij}]$ such that

$$\left[\prod_{\substack{1 \leq i,j \leq N \\ i \neq j}} x_{ij}^{k_{ij}} \right] = \left[\sum_{B \in \mathcal{B}[X]} \theta_B \prod_{(i,j) \in B} x_{ij}^{2g} \right].$$

So

$$\prod_{\substack{1 \le i,j \le N \\ i \ne j}} c_1(L_{ij})^{k_{ij}} = \pi \left(\left[\sum_{B \in \mathcal{B}[X]} \theta_B \prod_{(i,j) \in B} x_{ij}^{2g} \right] \right)$$

$$= \sum_{B \in \mathcal{B}[X]} \pi(\theta_B) \prod_{(i,j) \in B} c_1(L_{ij})^{2g},$$

which vanishes by Lemma 4.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

References

[1] M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A*, 308(1505):523–615, 1983.

[2] V. Yu. Baranovskiĭ. The cohomology ring of the moduli space of stable bundles with odd determinant. *Izv. Ross. Akad. Nauk Ser. Mat.*, 58(4):204–210, 1994.

[3] Peter Bouwknegt, Varghese Mathai, and Siye Wu. Bundle gerbes and moduli spaces. *J. Geom. Phys.*, 62(1):1–10, 2012.

[4] Michel Brion and Ivan Kausz. Vanishing of top equivariant Chern classes of regular embeddings. *Asian J. Math.*, 9(4):489–496, 2005.

[5] S. K. Donaldson. Gluing techniques in the cohomology of moduli spaces. pages 137–170, 1993.

[6] Richard Earl. The Mumford relations and the moduli of rank three stable bundles. *Compositio Mathematica*, 109(1):13–48, 1997.

[7] Richard Earl and Frances Kirwan. The Pontryagin rings of moduli spaces of arbitrary rank holomorphic bundles over a Riemann surface. *Journal of the London Mathematical Society*, 60:835–846, 12 1999.

[8] Richard Earl and Frances Kirwan. Complete sets of relations in the cohomology rings of moduli spaces of holomorphic bundles and parabolic bundles over a Riemann surface. *Proc. London Math. Soc. (3)*, 89(3):570–622, 2004.

[9] D. Gieseker. A degeneration of the moduli space of stable bundles. *J. Differential Geom.*, 19(1):173–206, 1984.

[10] R. Herrera and S. Salamon. Intersection numbers on moduli spaces and symmetries of a Verlinde formula. *Comm. Math. Phys.*, 188(3):521–534, 1997.

[11] Lisa Jeffrey, Young-Hoon Kiem, Frances C. Kirwan, and Jonathan Woolf. Intersection pairings on singular moduli spaces of bundles over a Riemann surface and their partial desingularisations. *Transform. Groups*, 11(3):439–494, 2006.

[12] Lisa Jeffrey and Jonathan Weitsman. Toric structures on the moduli space of flat connections on a Riemann surface. II. Inductive decomposition of the moduli space. *Math. Ann.*, 307(1):93–108, 1997.

[13] Lisa C. Jeffrey. Extended moduli spaces of flat connections on Riemann surfaces. *Math. Ann.*, 298(4):667–692, 1994.

[14] Lisa C. Jeffrey. The Verlinde formula for parabolic bundles. *J. London Math. Soc. (2)*, 63(3):754–768, 2001.

[15] Lisa C. Jeffrey and Frances C. Kirwan. Intersection pairings in moduli spaces of holomorphic bundles on a Riemann surface. *Electron. Res. Announc. Amer. Math. Soc.*, 1(2):57–71 (electronic), 1995.

[16] Lisa C. Jeffrey and Frances C. Kirwan. Intersection theory on moduli spaces of holomorphic bundles of arbitrary rank on a Riemann surface. *Ann. of Math. (2)*, 148(1):109–196, 1998.

[17] Lisa C. Jeffrey and Jonathan Weitsman. Symplectic geometry of the moduli space of flat connections on a Riemann surface: inductive decompositions and vanishing theorems. *Canad. J. Math.*, 52(3):582–612, 2000.

[18] Young-Hoon Kiem. Intersection cohomology of representation spaces of surface groups. *Internat. J. Math.*, 17(2):169–182, 2006.

[19] Young-Hoon Kiem and Jun Li. Vanishing of the top Chern classes of the moduli of vector bundles. *J. Differential Geom.*, 76(1):45–115, 2007.

[20] A. D. King and P. E. Newstead. On the cohomology ring of the moduli space of rank 2 vector bundles on a curve. *Topology*, 37(2):407–418, 1998.

[21] Frances Kirwan. On the homology of compactifications of moduli spaces of vector bundles over a Riemann surface. *Proc. London Math. Soc. (3)*, 53(2):237–266, 1986.

[22] Frances Kirwan. The cohomology rings of moduli spaces of bundles over Riemann surfaces. *J. Amer. Math. Soc.*, 5(4):853–906, 1992.

[23] Alina Marian. On the intersection theory of Quot schemes and moduli of bundles with sections. *J. Reine Angew. Math.*, 610:13–27, 2007.

[24] Alina Marian and Dragos Oprea. Counts of maps to Grassmannians and intersections on the moduli space of bundles. *J. Differential Geom.*, 76(1):155–175, 2007.

[25] E. Meinrenken. Witten's formulas for intersection pairings on moduli spaces of flat G-bundles. *Adv. Math.*, 197(1):140–197, 2005.

[26] M. S. Narasimhan and S. Ramanan. Deformations of the moduli space of vector bundles over an algebraic curve. *Ann. Math. (2)*, 101:391–417, 1975.

[27] P. E. Newstead. Characteristic classes of stable bundles of rank 2 over an algebraic curve. *Trans. Amer. Math. Soc.*, 169:337–345, 1972.

[28] S. Ramanan. The moduli spaces of vector bundles over an algebraic curve. *Math. Ann.*, 200:69–84, 1973.

[29] C. S. Seshadri. Degenerations of the moduli spaces of vector bundles on curves. In *School on Algebraic Geometry (Trieste, 1999)*, volume 1 of *ICTP Lect. Notes*, pages 205–265. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2000.

[30] Bernd Siebert and Gang Tian. Recursive relations for the cohomology ring of moduli spaces of stable bundles. *Turkish J. Math.*, 19(2):131–144, 1995.

[31] Adam S. Sikora. Character varieties. *Trans. Amer. Math. Soc.*, 364(10):5173–5208, 2012.

[32] Constantin Teleman and Christopher T. Woodward. The index formula for the moduli of G-bundles on a curve. *Ann. of Math. (2)*, 170(2):495–527, 2009.

[33] Michael Thaddeus. Conformal field theory and the cohomology of the moduli space of stable bundles. *J. Differential Geom.*, 35(1):131–149, 1992.

[34] Jonathan Weitsman. Geometry of the intersection ring of the moduli space of flat connections and the conjectures of Newstead and Witten. *Topology*, 37:115–132, 1998.

[35] Edward Witten. On quantum gauge theories in two dimensions. *Comm. Math. Phys.*, 141(1):153–209, 1991.

[36] Takahiko Yoshida. The generating function for certain cohomology intersection pairings of the moduli space of flat connections. *J. Math. Sci. Univ. Tokyo*, 8(3):541–558, 2001.

[37] Don Zagier. On the cohomology of moduli spaces of rank two vector bundles over curves. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 533–563. Birkhäuser Boston, Boston, MA, 1995.